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Tunneling Transitions in Quantum Mechanics, Field Theory and Gravity.

Tunneling Transitions in Quantum Mechanics, Field Theory and Gravity

Eternal inflation and bubble collisions

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Abstract

In this essay, the implications of quantum tunneling on a scalar field with two distinct minima are presented. The analysis generalizes the WKB approximation for N-dimensions, and follows previous work from Coleman and De Luccia. We find that the tunneling process leads to the nucleation of true vacuum bubbles expanding at the speed of light inside the false vacuum background. When gravity is considered, not only the spacetime affects the creation of bubbles, but also the bubbles affect the shape of the spacetime. The change in vacua creates different expansion rates inside and outside the bubble. After tunneling takes place, outside the bubble there is a de Sitter spacetime, while inside it creates an open FRW universe. Depending on the rate of expansion of the false vacuum, there will be collisions of different bubbles. In the last section, we calculate that the number of expected collisions for a universe like ours is $N \propto (H_i H_f)^{-2}$, being $H_{i,f}$ the Hubble parameter inside and outside the bubble, respectively.

Contents

1 Introduction

In this essay, I will explain how a straight forward concept as Quantum Tunneling can lead to big implications on our universe. For this, both Quantum Mechanics and General Relativity will be combined throughout the calculations. This will take us to possible direct evidence of fundamental theories which predict eternal inflation, such as String Theory.

However, we must start from the beginning. It is well known that in classical mechanics a system will always follow the path of least action [1]. From there, we can get the equations of motion (e.o.m.), which are used to study the evolution of the system after long periods of time.
In physics, a fixed point is the value for which the system remains in equilibrium, and it is given where the potential has vanishing slope. However, the stability of these fixed points depends on how the system reacts to small perturbations. For instance, every minimum of the function will be stable, while all the maxima will be unstable. Moreover, to go from one minimum to another one, the system requires having enough energy to climb up the potential barrier separating them both.

Even though this picture seems intuitively right, at the beginning of last century it was shown to be incomplete. With the development of Quantum Mechanics, it was discovered what we now call Quantum tunneling [2]. In this process, a state can travel through a potential barrier which is more energetic than the system itself. This is forbidden in classical mechanics, and the probability of this happening depends on the shape of the potential barrier.

Therefore, I will start by introducing the mathematical techniques used to calculate the tunneling rate in N-Dimensions, called the *WKB approximation*. This method works under the assumption that the potential varies slowly. By taking the infinite limit of dimensions, this technique can be applied to Quantum Field Theory.
The most important aspect from Quantum Field Theory is that it can be thought as an infinite number of coupled harmonic oscillators [3]. Therefore, tunneling at one point of space can affect its surroundings. We will work with a free scalar field in four-dimensional spacetime with nonderivative interactions:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi), \tag{1.1}$$

where $\mathcal{L}$ is the Lagrangian density, and $U(\phi)$ posses two relative minima, $\phi_\pm$, where only ϕ_- is an absolute minimum. Because of tunneling, we expect a system starting at the false vacuum, ϕ_+, to end up at the true vacuum, ϕ_-. Therefore, the only stable fixed point of a quantum system will be the global minimum. Moreover, the nucleated region will expand throughout the field, creating a bubble of true vacuum inside the false vacuum background.

However, any physical theory that does not consider the effect of gravity will be incomplete. Even though its effect might seem negligible in this system, it does make a big difference on the creation and evolution of these bubbles. This is because of the effect of a cosmological constant on the Universe. As we know, different vacuum energies provide different geometries on our spacetime, and so lead to an expanding or contracting universe inside our bubble, depending on the sign of the minimum.
The results presented in this essay are predictions of the eternal inflationary model. In the same way that we can study the probability of having a bubble nucleated, we can further study the probability of two different bubbles colliding.

Section 2 discusses the general formalism for studying barrier-penetration using the WKB approximation. This method will be applied to QFT in Section 3, where the evolution equation will be solved for the limit of small energy-density between the two vacua. In Section 4, gravity will be included in the system, influencing the geometry of the different vacuum regions. These results will be used to calculate the properties of tunneling inside a de Sitter universe. In section 5, I will end up talking about the evidence we would expect to find if we lived inside a nucleated bubble.

2 Tunneling in Quantum Mechanics

As the main focus of this essay is to apply the tunneling effect to QFT, we will introduce the WKB approximation for N-dimensional quantum mechanics. For that, consider a particle of energy E and mass $m = 1$ living in a N-dimensional space under the presence of a potential, $V(\mathbf{q})$. The particle is described by its wavefunction, $\psi(\mathbf{q})$, satisfying the time independent Schrödinger equation,

$$-\frac{\hbar^2}{2}\vec{\nabla}^2\psi + V(\mathbf{q})\psi = E\psi \tag{2.1}$$

where $\vec{\nabla}$ is the derivative with respect to each direction in space and $\hbar$ is the reduced Plank constant. For complex potentials, such as the shown in Fig.1, it is very difficult to find an exact equation for the wavefunction of this particle. However, advances were made by considering the limit for slow varying $V(\mathbf{q})$ [2]. With that, we expect the wavefunction to have a general form:

$$\psi(\mathbf{q})\ \alpha e^{-\frac{u(\mathbf{q})}{\hbar}} \tag{2.2}$$

where $u(\mathbf{q})$ can be found by solving the Schrödinger equation. Taking an infinite expansion of terms such as

$$u(\mathbf{q}) = u_o(\mathbf{q}) + \hbar u_1(\mathbf{q}) + \mathcal{O}(\hbar^2) \tag{2.3}$$

where u_o relates to a free particle, which has $u_o'' = 0$. In a square potential barrier case, this would be the final solution. However, for a varying potential, the next terms can be found by using perturbation methods and requiring each order in $\hbar$ to satisfy Eq. (2.1).

Thus, we get the following WKB equations for N-dimensions:

$$\vec{\nabla} u_o \vec{\nabla} u_o = 2(V - E) \tag{2.4}$$

$$\vec{\nabla} u_o \vec{\nabla} u_1 = \frac{1}{2} \vec{\nabla}^2 u_o \tag{2.5}$$

which are the equations that we must solve to find an approximation of the evolution of the system, up to $\mathcal{O}(\hbar^2)$.

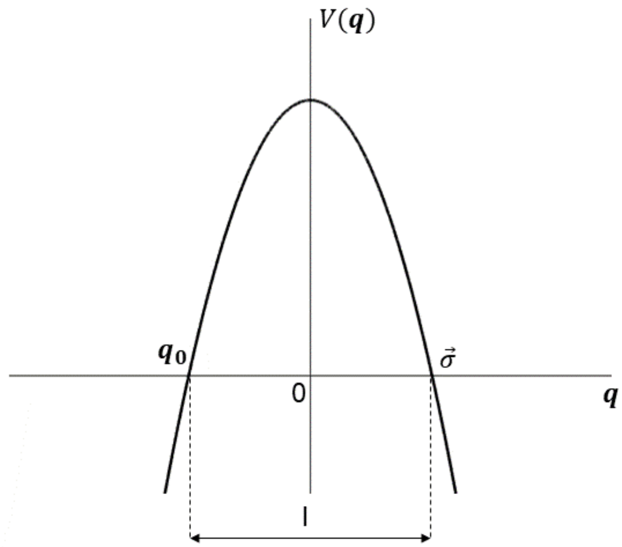

Figure 1: General potential with higher energy than the system itself. The region **I** is classically forbidden, making the tunneling process happen from $\mathbf{q}_o$ to $\vec{\sigma}$.

In the simple case where N=1, we would proceeded by solving these equations and implying continuity, what would give us the wavefunction. This should be straight forward, as there is only one possible path to follow, given that the particle travels from left to right in a 1D space. However, in the N>1 case, there are multiple paths with the same boundary terms. This establishes an ambiguity on the direction of $\vec{\nabla}$, making the calculation very hard to solve.

For this, a further approximation was taken by Banks, Bender and Wu [4], where they introduced the Most Probable Escape Path (MPEP). Instead of considering the weight of all the possible paths by using standard path integral techniques, the measure is dominated by the contribution of a given path. In the allowed region, this path will correspond to the classical solution of the e.o.m., but in the forbidden region it will correspond to the MPEP. This reduces considerably the problem's complexity, as we just need to find this path and integrate over it.

For this derivation, I will follow the reference [5]. We will start the seek of the MPEP by considering a curve $\mathbf{Q}(\lambda) \in \mathbb{R}^N$. By defining a vector space around the curve, we find that it has one tangent vector

$$\mathbf{v}_{\parallel}(\lambda) = \frac{\partial \mathbf{Q}}{\partial \lambda} \tag{2.6}$$

and N-1 perpendicular vectors

$$\mathbf{v}_{\perp}^{i}(\lambda) \tag{2.7}$$

such that:

$$\mathbf{v}_{\parallel} \cdot \mathbf{v}_{\perp}^{i} = 0; \qquad \mathbf{v}_{\perp}^{i} \cdot \mathbf{v}_{\perp}^{j} = \delta_{ij} \qquad \text{for i,j } \in 1,2,...,\text{N-1} \tag{2.8}$$

Therefore, we can express the differential operator in the curve as:

$$\vec{\nabla}|_{\mathbf{q}=\mathbf{Q}} = \frac{\mathbf{v}_{\parallel}}{\left|\mathbf{v}_{\parallel}\right|^2}(\mathbf{v}_{\parallel} \cdot \vec{\nabla})|_{\mathbf{q}=\mathbf{Q}} + \sum_{i=1}^{N-1} \frac{\mathbf{v}_{\perp}^{i}}{|\mathbf{v}_{\perp}^{i}|^2}(\mathbf{v}_{\perp}^{i} \cdot \vec{\nabla})|_{\mathbf{q}=\mathbf{Q}} \tag{2.9}$$

which is just the projection of the differential operator on each direction of the vector space. But so far, we have not made any approximation in order of finding the MPEP, for that we will apply the following constraints

$$\mathbf{v}_{\perp}^{i} \cdot \vec{\nabla} u|_{\mathbf{q}=\mathbf{Q}} = 0 \qquad \text{for } i = 1, 2, ..., N-1 \tag{2.10}$$

which defines the path to vary only in the tangent direction to $\mathbf{Q}(\lambda)$. Then, redefining the differential operator to just act in that direction, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}s} = \frac{\mathbf{v}_{\parallel}}{\left|\mathbf{v}_{\parallel}\right|} \cdot \vec{\nabla}|_{\mathbf{q}=\mathbf{Q}} \tag{2.11}$$

$$\mathrm{d}s = |\mathrm{d}\mathbf{Q}| = \sqrt{\frac{\mathrm{d}Q}{\mathrm{d}\lambda} \cdot \frac{\mathrm{d}Q}{\mathrm{d}\lambda}}\mathrm{d}\lambda = \left|\mathbf{v}_{\parallel}\right|\mathrm{d}\lambda. \tag{2.12}$$

This solves the ambiguity mentioned before, as now we already have one preferred path to integrate over. Therefore, the WKB equations (Eq. (2.4,2.5)) imply:

$$u_o = \pm \int^s \mathrm{d}s\, \kappa(s) \tag{2.13}$$

$$u_1 = \int^s \mathrm{d}s\, \frac{\kappa'(s)}{2\kappa(s)} = \frac{1}{2}\ln(\kappa) \tag{2.14}$$

where

$$\kappa = \sqrt{2(V(\mathbf{Q}) - E)}. \tag{2.15}$$

Considering both solutions, the wavefunction will have the form:

$$\psi(q) \approx \frac{1}{[|2(V(\mathbf{q}) - E)|]^{\frac{1}{4}}} \left[\alpha_+ e^{\frac{1}{\hbar}\int^s \mathrm{d}s\,\kappa(\mathbf{q})} + \alpha_- e^{-\frac{1}{\hbar}\int^s \mathrm{d}s\,\kappa(\mathbf{q})}\right], \tag{2.16}$$

where $\alpha_{\pm}$ are arbitrary constants which will be defined to satisfy the boundary conditions. For instance, given a particle travelling from right to left, we find $\alpha_{+} = 0$ and α_{-} to be a normalising factor. In the N>1 case, the integral is evaluated through the MPEP. But we have not defined that path yet; for that, we will minimise u_o by making small fluctuations in the directions normal to $\mathbf{Q}$, and then setting the difference to zero. After some algebra, we find the expression

$$\frac{\mathrm{d}^2\mathbf{Q}}{\mathrm{d}\lambda^2} - \vec{\nabla}V = 0 \tag{2.17}$$

where λ is chosen to satisfy

$$\left(\frac{\mathrm{d}s}{\mathrm{d}\lambda}\right)^2 = 2(V - E). \tag{2.18}$$

Therefore, we see that λ plays the role of the imaginary time. Similarly, for the classically allowed region we find

$$\frac{\mathrm{d}^2\mathbf{Q}}{\mathrm{d}t^2} + \vec{\nabla}V = 0, \tag{2.19}$$

which are just the e.o.m. of the system. Therefore, the MPEP corresponds to the e.o.m. with imaginary time in the classically forbidden region. With this definition of the path, we can now use our expression of ψ to describe the evolution of our system.

Studying the Eq. (2.16), we see that in the classically allowed region ($E > V$), the solution is an oscillating function; while in the classically forbidden region ($V > E$), there will be an exponentially decaying function with growing $|\mathbf{q}|$. These two regions will therefore be well defined, but we must study how the solution behaves at the turning points from the forbidden to the allowed region and vice versa.

The standard procedure consists on approximating the potential at the boundary to a linear function [2], and so we can write

$$V(\mathbf{q}) - E \approx g(\mathbf{q} - \mathbf{a}) \tag{2.20}$$

where g is a general constant and the turning point is located at $\mathbf{q} = \mathbf{a}$. By requiring the wavefunction to be continuous and differentiable, we find the transmission rate to be:

$$\Gamma = Ae^{-B/\hbar}\,[1 + \mathcal{O}(\hbar)], \tag{2.21}$$

where

$$B = 2\int^{s} \mathrm{d}s\sqrt{2(V - E)}. \tag{2.22}$$

with A being a normalising constant. We can tell that this expression simplifies to the square barrier penetration for constant $V(q)$ inside the forbidden region. Moreover, we find that any kind of barrier penetration is exponentially suppressed. However, this is not a very convenient expression to work with, given that solving this integral for general potentials might take a lot of algebra. For this, in their original paper, Coleman [6] derived what he called the *Bounce*.

2.1 The Bounce

From Eq. (2.21), there are two important coefficients that will define the transition rate of our system, A and B. Coleman and Callan dedicated one article to the calculation of the coefficient A [7]. However, given that these transition rates will be applied to gravity, I won't focus on this aspect of their research. This computation would involve the evaluation of a functional determinant, leading to the nonrenormalizability of the theory. Therefore, in this section I will follow their work on how to express the B coefficient in a more convenient way.

The first thing that we must do, is to find the constraints that the MPEP applies in the system, and so try to solve these equations for B. From Eq. (2.17), we know that the MPEP is given by the path that satisfies the e.o.m. with imaginary time (τ). Therefore, given the general potential in Fig.1, we have the following equations to follow:

$$\frac{\mathrm{d}^2\mathbf{q}}{\mathrm{d}\tau^2} = \frac{\partial V}{\partial \mathbf{q}} \tag{2.23}$$

with

$$\frac{1}{2}\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}\tau}\cdot\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}\tau} - V = -E. \tag{2.24}$$

As these equations are just an imaginary time version of the e.o.m., we can just describe the system with the Euclidean Lagrangian L_E,

$$L_E = \frac{1}{2}\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}\tau}\cdot\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}\tau} + V. \tag{2.25}$$

Now, defining the boundary terms as in Fig.1, the system begins at rest at $\mathbf{q} = \mathbf{q}_o$ and it tunnels all the way up to $\mathbf{q} = \vec{\sigma}$. By solving the WKB equations, we find that the starting point, $\mathbf{q}_o$, can only be reached asymptotically, as τ goes to minus infinity.

Provided that L_E is explicitly independent of time, there is time invariance. This lets us choose, without loss of generality, the imaginary time at which the tunneling takes place to be $\tau = 0$. Given that the energy must be conserved through the tunneling and $V(\mathbf{q}_o) = V(\sigma) = E$, the following constraints apply:

$$\lim_{\tau\to-\infty} \mathbf{q} = \mathbf{q}_o \qquad\qquad \left.\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}\tau}\right|_o = \vec{0}. \tag{2.26}$$

From the MPEP we have that $\mathrm{d}s = \mathrm{d}\mathbf{q}$ and Eq. (2.24) implies

$$\mathrm{d}\mathbf{q} = \sqrt{2(V-E)}\mathrm{d}\tau.$$

Then, by direct substitution in Eq. (2.22) we obtain

$$B = 2\int_{\mathbf{q}_o}^{\vec{\sigma}} \mathrm{d}s\,\sqrt{2(V-E)} \tag{2.27}$$

$$= 2\int_{-\infty}^{\vec{0}} \mathrm{d}\tau\,(2(V-E)) \tag{2.28}$$

$$= 2\int_{-\infty}^{\vec{0}} \mathrm{d}\tau\,[L_E - E], \tag{2.29}$$

where in the last line I used Eq. (2.24) with the definition of L_E (Eq. (2.25)). Moreover, we can tell that the equations of motion for positive τ are symmetrical to the ones we just derived. Therefore, we expect the system to go back to $\mathbf{q} = \mathbf{q}_o$ in the limit where τ goes to infinity.

With this, we define the full range integral as the *Bounce*. As we have a factor of 2 in front of our tunneling rate, we can interpret B as being the the total Euclidean action for the bounce,

$$B = \int_{-\infty}^{\infty} \mathrm{d}\tau\,[L_E - E] \equiv S_E - S_E(\mathbf{q}_o), \tag{2.30}$$

where I have used the fact that the integral of the energy corresponds to the Euclidean action of the stationary solution in which the particle does not tunnel. Therefore, to find the coefficient B, we just need to find the bounce, which is the solution of the imaginary-time e.o.m. obeying the constraints from Eq. (2.26). In the case where there are multiple bounces solutions, the preferred will be given by the one with lower Euclidean action. This definition of the bouce will be used in the next section to calculate the tunneling rate in a quantum field.

3 Tunneling in Quantum Field Theory

To generalise the results of the previous section to standard scalar quantum field theory, first we need to prove that both theories are equivalent. For this, I will follow the ideas from [5][8]. The best way of relating a quantum field to quantum mechanics is by studying the action of a standard scalar field, $\phi(t, \vec{x})$, evolving under the influence of a general potential, $U(\psi)$. This is described by the action

$$S = \int \mathrm{d}t \int \mathrm{d}^3x \left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - U(\phi)\right] \tag{3.1}$$

where $\dot{} = \frac{\partial}{\partial t}$ and $\vec{\nabla}$ is the space derivative in each direction. From this action, we can compare the multidimensional Hamiltonin in quantum dynamics with the Hamiltonian of the field:

$$H = \int \mathrm{d}^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + U(\phi)\right] \tag{3.2}$$

where the conjugate momentum is given by $\pi = \dot{\phi}$. As we know, the integral can be thought as an infinite sum of the integrand, weighted by the measure. Moreover, we can describe the field as

$$\left\{\phi(t, \vec{x} \in \mathbb{R}^N)\right\} = \left\{\phi(t, \vec{x}_1), \phi(t, \vec{x}_2), ...\right\}, \tag{3.3}$$

which means that each point of the field can be thought to be a further three dimensions in our quantum mechanical system. Given that we have infinite points in the space, we can define this system as an infinite dimension quantum mechanics. In this sense, we see that we should make the following generalisations:

$$\frac{1}{2}\vec{p}\cdot\vec{p} \quad \Longrightarrow \quad \int \mathrm{d}^3x\, \frac{1}{2}\pi^2 \tag{3.4}$$

$$V(\mathbf{q}) \quad \Longrightarrow \quad V[\phi] = \int \mathrm{d}^3x \left[\frac{1}{2}(\vec{\nabla}\phi)^2 + U(\phi)\right], \tag{3.5}$$

where the generalised potential, $V[\phi]$, will be the one that determines the tunneling rate of the system. This makes the problem to be less intuitive, given that the potential now depends on the configuration of the whole space. This is what couples every point in our spacetime with each other. Now I will calculate the tunneling rate of a scalar field, and how a tunneled point influences its surroundings. I will closely follow [6].

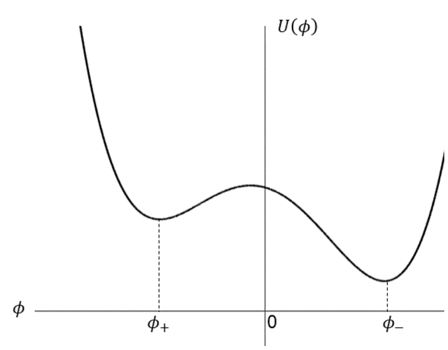

Figure 2: General potential with two minima. Being ϕ_- the true vacuum.

To apply these results to a quantum field, first we must make the substitutions introduced above. Let this field to evolve under the effect of a general potential shown in Fig.2. The whole field starts at rest in the false vacuum, $\phi = \phi_+$, and we expect it to tunnel down to the true vacuum, ϕ_-. Its Euclidean e.o.m. will be given by

$$\left(\frac{\partial^2}{\partial\tau^2} + \nabla^2\right)\phi = U'(\phi) \tag{3.6}$$

where the boundary conditions are

$$\lim_{\tau\to\pm\infty}\phi(\tau,\vec{x}) = \phi_+ \qquad\qquad \frac{\mathrm{d}\phi}{\mathrm{d}\tau}(0,\vec{x}) = 0 \tag{3.7}$$

Therefore, we conclude that the bounce is given by:

$$B \equiv S_E - S_E(\phi_+) = \int \mathrm{d}\tau\,\mathrm{d}^3x\left[\frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + U(\phi) - U(\phi_+)\right] \tag{3.8}$$

and so, for B to be finite, we need a further constraint

$$\lim_{|\vec{x}|\to\infty}\phi(\tau,\vec{x}) = \phi_+. \tag{3.9}$$

This is a very nice condition, given that now both the boundaries and e.o.m. are invariant under four-dimensional Euclidean rotations, corresponding to a SO(4) symmetry. By defining ρ as

$$\rho = \sqrt{\tau^2 + |\vec{x}|^2}, \tag{3.10}$$

the previous equations can be written as

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\rho^2} + \frac{3}{\rho}\frac{\mathrm{d}\phi}{\mathrm{d}\rho} = U'(\phi) \qquad\qquad \lim_{\rho\to\infty}\phi(\rho) = \phi_+. \tag{3.11}$$

$$B = S_E - S_E(\phi_+) = 2\pi^2\int_0^\infty \rho^3\,\mathrm{d}\rho\left[\frac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\rho}\right)^2 + U - U(\phi_+)\right] \tag{3.12}$$

where for the e.o.m. and the bounce the differential operators have been expressed in 4-dimensional polar coordinates. The $2\pi^2$ factor in the bounce comes from integrating out the angular coordinates.

However, before calculating the bounce, let me stop to revise the general picture so far. We have discovered that tunneling will just occur for systems which can be described by the bounce solution. In the coordinate system we are working with, τ runs over the positive numbers, which will treat the problem as being in *time reverse*. Therefore, it starts at imaginary time $\tau = 0$ at some initial value $\phi = \sigma$, which in reality corresponds to the endpoint of the tunneling. From there, when τ grows all the way up to infinity, ϕ reaches asymptotically the false vacuum ϕ_+.

The Eq.(3.12) can be interpreted as a free standard action with some viscous damping. This standard action would correspond to the motion of a particle under the presence of a potential given by $-U(\phi)$, such as in Fig.3. With this, we can study for which initial conditions the system will bounce to ϕ_+ when $\rho \to \infty$. Even though the exact solution for the allowed tunnels is model dependant, we can still show that for a general system there will always be a bounce solution [6]. Therefore, a particle under the influence of the potential $-U$ can behave in one of the following ways depending on the initial conditions:

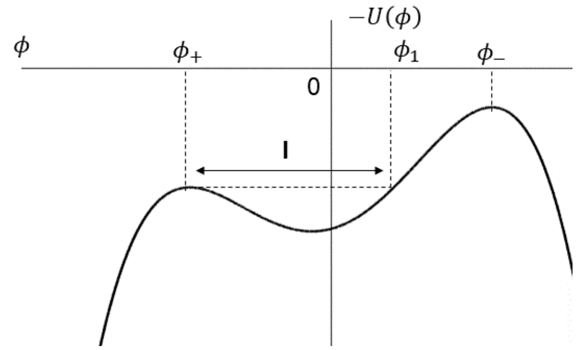

Figure 3: The potential energy for the mechanical analogy to Eq. (3.11), corresponding to $-U(\phi)$

- **Undershoot:** In this case, the particle is not able to climb the potential up to ϕ_+. Demonstrating undershoot is trivial, for instance all the particles released between ϕ_1 and ϕ_+ (Region **I** in Fig.3) won't have enough energy to climb up the hill. As the damping term always diminishes the energy, there will be some values at the right of ϕ_1 which will also undershoot.

- **Overshoot:** These are the initial values for which the particle will eventually reach ϕ_+. To demonstrate overshoot, its useful to work near the true vacuum, ϕ_-. For values of ϕ very close to the true vacuum, we can linearise Eq. (3.11) to:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \frac{3}{\rho} - \mu^2\right)(\phi - \phi_-) = 0, \tag{3.13}$$

  where $\mu^2 \equiv U''(\phi_-)$.
  We can choose ϕ to be initially sufficiently close to ϕ_- such that it will stay arbitrarily close to ϕ_- for arbitrarily large ρ. However, from the equation above we see that for sufficiently large ρ, the damping term can be neglected, and so we recover a classical free particle under the effect of the potential $-U(\phi)$. As the energy at ϕ_- is higher than in ϕ_+, the particle will be able to climb all the way up the hill, and so overshoot.

Given that both regions must exist for any generic potential with two minima, by continuity, there must be a midpoint between ϕ_1 and ϕ_- for which particles will start overshooting. As stated before, when we have overshoot it means that a tunneling from ϕ_+ to σ will eventually occur. However, finding the exact solution of the bounce for a generic potential is not straightforward. Therefore, in the next section, I will introduce a further approximation to describe the evolution of the field during and after tunneling.

3.1 Thin-wall Approximation

This approximation expects the field to be constant both before and after the overshoot takes place. Therefore, the tunneling process happens very quickly, at a given $\rho = R$. This value of ρ will be where the field goes from the false vacuum to the true vacuum, making the bounce much simpler to solve.

As we expect the field to be constant elsewhere, the description of the field evolution will be:

$$\phi = \begin{cases} -a & \text{for } \rho \ll R \\ \phi_s(\rho - R) & \text{for } \rho \approx R \\ a & \text{for } \rho \gg R \end{cases} \tag{3.14}$$

where $U(-a) = -\varepsilon \ll 1$ (ε is assumed to be small for this approximation to be valid) and $U(a) = 0$. The function $\phi_s(\rho)$ is a one dimensional soliton, which is the best description for the variation of ϕ at the wall. A full definition of the soliton solution can be found in the first chapter of Weinberg's book [8].

We can tell that Eq. (3.14) is essentially a four-dimensional bubble of true vacuum (from the SO(4) symmetry), surrounded by a thin wall. The only thing left to do is to calculate the preferred $\rho = R$ for which nucleation will take place. This is described by the MPEP, so we must just solve the bounce integral:

$$B = 2\pi^2 \int_0^\infty \rho^3 \,\mathrm{d}\rho \left[\frac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\rho}\right)^2 + U(\phi) - U(\phi_+) \right] \tag{3.15}$$

$$= 2\pi^2 \left[\int_0^{R_-} \rho^3 \,\mathrm{d}\rho(-\varepsilon) + \int_{R_-}^{R_+} \rho^3 \,\mathrm{d}\rho \left[\frac{1}{2}\left(\frac{\mathrm{d}\phi_s}{\mathrm{d}\rho}\right)^2 + U \right] + \int_{R_+}^\infty \rho^3 \,\mathrm{d}\rho(0) \right] \tag{3.16}$$

$$= -\frac{1}{2}\pi^2 R^4 \varepsilon + 2\pi^2 R^3 S_s. \tag{3.17}$$

The first term is the action of a true vacuum bubble of radius R and the second term comes from the wall of this bubble, where S_s is the action of 1dimensional soliton. As the MPEP corresponds to the path of least action, now just have to vary it with respect to R to find the preferred tunneling value of ρ:

$$\frac{\mathrm{d}S_E}{\mathrm{d}R} = 0 = -2\pi^2 R^3 \varepsilon + 6\pi^2 R^2 S_s. \tag{3.18}$$

Hence,

$$R = \frac{3S_s}{\varepsilon} \tag{3.19}$$

where we can see that for very small ε, it tends to infinity. We can make sense of this by assuming that for $\varepsilon = 0$ there is no such a thing as true or false vacuum, and so tunneling between both vacua won't occur, given that both minima will be stable.

After tunneling, the system enters the classically allowed region. From the MPEP equations (Eq. (2.19)), it will evolve according to the classical field equation,

$$-\frac{-\partial^2\phi}{\partial t^2}+\nabla^2\phi=U'(\phi). \tag{3.20}$$

Moreover, as the Euclidean space is equivalent to Minkowski with imaginary time, we have

$$\phi(t=0,\vec{x})=\phi(\tau=0,\vec{x}), \qquad \frac{\partial}{\partial t}\phi(t=0,\vec{x})=0. \tag{3.21}$$

We can see that the same function, $\phi(\rho)$, that gives the shape of the bounce in four-dimensional Euclidean space also defines the system after the tunneling in ordinary 3+1-dimensional spacetime. The O(4) symmetry has transformed into O(3,1) symmetry.
Therefore, the four-dimensional bubble living in Euclidean space transforms into a three-dimensional bubble of true vacuum in the field space, where time plays an important role on its evolution. This is because the bubble solution is fully defined by the value of ρ, and so it must keep that definition after its materialization. Because of that, we have the following constraint:

$$\phi(t,\vec{x})=\phi(\rho=(|\vec{x}|^2-t^2)^{1/2}), \tag{3.22}$$

which comes from the fact that $t^2=-\tau^2$. Therefore, we have a full description of the evolution of the bubble by the condition

$$|\vec{x}|^2=R^2+t^2, \tag{3.23}$$

which is the general equation of a hyperbola. We expect R to be a microphysical number, and so once the bubble materializes, it begins to expand almost at the speed of light. An example of this evolution can be found in Fig.4.
During this expansion the energy of the system must be conserved, so the loss of energy inside of the bubble goes to the kinetic energy of the expanding wall.
This means that the only truly stable fixed point will be ϕ_-, and all the false vacuum of the field will eventually tunnel to the true vacuum, reaching the stable limit of the system.

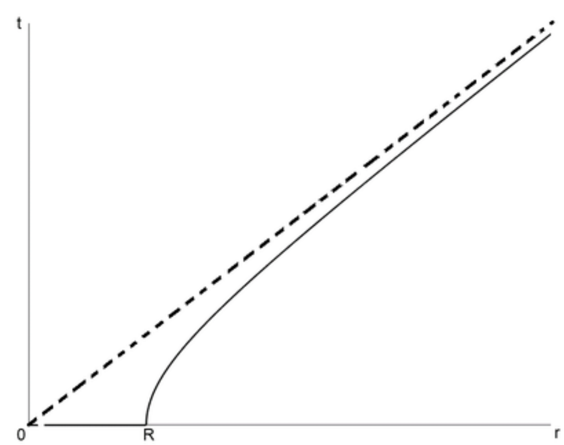

Figure 4: Evolution of the radius of a bubble with nucleation radius R. We see that its expansion describes an hyperbola, with a speed which tends asymptotically to the speed o light (dashed line)

However, it might seem a little bit ambiguous the fact that we can have different sizes of bubbles being nucleated. As stated before, there is a full range of overshoot solutions in a system, so there can be tunneling from the false vacuum to different field values at the other side of the hill, which will have a potential energy $U(\sigma)$. Each kind of tunneling solution will be given at a certain ρ, that will correspond to the radius of the bubble in the thin wall limit. As the energy during the tunneling must be conserved, the bubble must satisfy

$$\frac{4}{3}R^3U(\phi_+) = \frac{4}{3}R^3U(\sigma) + E_{\text{wall}}, \tag{3.24}$$

where E_{wall} is the energy stored in the wall.
As the bubble volume grows as R^3 and the wall as R^2, the final radius of the nucleated bubble will be given by the combination which satisfies the conservation of energy.
Therefore, we can have a wide variety of bubbles being nucleated, these will have different energy inside, and so different size. However, after the nucleation, the field inside the bubble will evolve accordingly to its potential U, and so it will oscillate around ϕ_- until it looses enough energy to stay at rest in the minimum. In Fig.5 there are two examples of two possible solutions to the same system.

In this section, we have derived how the B coefficient affects the nucleation and evolution of a true vacuum bubble in Minkowski space. However, any physical theory which does not contain gravity is incomplete. For that, in the next section, I will introduce how a curved spacetime affects the results found in this section.

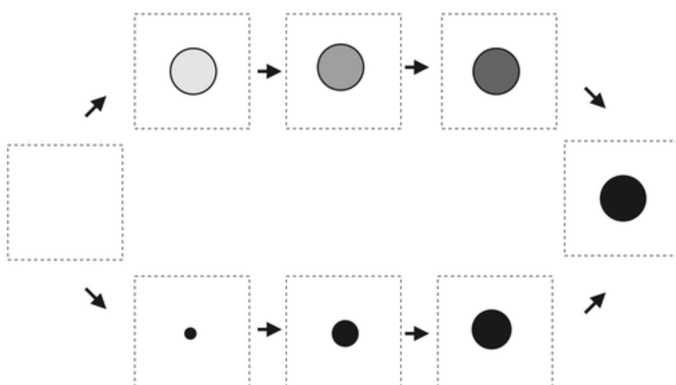

Figure 5: Two of the infinite number of paths through the potential barrier that connect the false vacuum, ϕ_+, with a configuration with the same energy on the other side of the hill, σ. In the upper path, the bubbles are all the same size, but the field progresses up to the true vacuum. In the lower path, the bubble is always on the true vacuum but expands. Credits:12$^{\text{th}}$ Chapter [8]

4 The addition of Gravity

As one might expect, the vacuum decay takes place on scales at which gravitational effects are negligible. However, once the bubble is nucleated, it will expand inside a universe, and its speed will be affected by the geometry of the spacetime. Moreover, not only does gravitation affect vacuum decay, but also vacuum decay affects gravitation.

In Quantum Field Theory, we only care about differences of energy, given that a constant term in the action does not affect the equations of motion. However, in gravity this is not the case, for instance the scalar field action now has the form:

$$S = \int \mathrm{d}^4x\,\sqrt{-\mathrm{g}}\left[\frac{1}{2}\mathrm{g}_{\mu\nu}\partial^\mu\phi\partial^\nu\phi - U(\phi) - \frac{1}{2\kappa}R\right], \tag{4.1}$$

where $\mathrm{g}_{\mu\nu}$ is the spacetime metric, $\kappa = 8\pi G$ and R is the curvature scalar. When gravity is considered, adding a constant is like adding a term proportional to $\sqrt{-\mathrm{g}}$ to the Lagrangian, which will have the effect of a cosmological constant.
It is well known that a scalar field resting at its minimum will create a cosmological constant in the universe, and depending on its sign, will form a de Sitter(dS) or anti-de Sitter(AdS) geometry. This means that the universe does care about the absolute value of the energy. Thus, once the vacuum decays, the gravitational theory changes inside the bubble, and so the cosmological constant inside will be different to the one outside, creating different geometries at both regions.

In this section, I will closely follow the calculations from Coleman and de Luccia[9]. The method and system used will be the same as for QFT, but with few further terms inside the action, which will give the gravitational effects to the solution.

We start by defining the metric of the Euclidean space provided by the false vacuum. For this, we know that a cosmological constant preserves rotational symmetry, which will make all the calculations much simpler to solve. Then, we define a radial curve to be a curve of fixed angular coordinates, and so to be normal to the three-spheres through which it passes. To measure distance along these radial curves we will choose the coordinate ξ, so the element of length is of the form

$$(ds)^2 = (d\xi)^2 + \rho(\xi)^2(d\Omega_3)^2 \tag{4.2}$$

where $(d\Omega_3)$ is the element of distance on a unit three-sphere and ρ gives the radius of curvature of each three-sphere. In flat space, $\rho = \xi$, so the radius of curvature is equivalent to the radial curve. From this metric we can calculate the curvature scalar, and so introduce it to the action stated above. As the procedure for calculating the bounce is the same as in flat spacetime, we know that the bounce will depend on the Euclidean action of the system, and so

$$S_E = 2\pi^2 \int \mathrm{d}\xi \left(\rho^3\left(\frac{1}{2}\phi'^2 + U\right) + \frac{3}{\kappa}(\rho^2\rho'' + \rho\rho'^2 - \rho)\right). \tag{4.3}$$

To calculate the MPEP we must use the Euler-Lagrange equations, which gives the following e.o.m.

$$\phi'' + \frac{3\rho'}{\rho}\phi' = \frac{\mathrm{d}U}{\mathrm{d}\phi}, \tag{4.4}$$

where $' = \frac{\mathrm{d}}{\mathrm{d}\xi}$. This equation only differs from the flat space case in two things. First, the independent variable is now ξ instead of ρ, which will be important for the growth of the bubble after materialization. Secondly, the coefficient of the ϕ' term involves a ρ'/ρ, instead of $1/\rho$. However, this is a trivial chance, since in the thin-wall approximation we neglect this term, given that the field is constant at both vacuum regions. Moreover, solving the Einstein's Equation we find that

$$\rho'^2 = 1 + \frac{1}{3}\kappa\rho^2\left(\frac{1}{2}\phi'^2 - U\right). \tag{4.5}$$

As we can see, gravity maintains the SO(4) symmetry of the bounce throughout the derivation. In last section, we found that this fact implied the materialization of a true vacuum bubble after the tunneling. Thus, we shall also expect the creation of a bubble in this case, whose properties can be calculated via the thin-wall approximation.

4.1 Thin-wall with gravity

As in the flat space case, this approximation expects the field to be constant both before and after the tunneling takes place, at $\rho = \bar{\rho}$. Therefore, the field can be divided into the following three regions:

$$\phi = \begin{cases} \phi_- & \text{for } \rho \ll \bar{\rho} \\ \phi_s(\rho - R) & \text{for } \rho \approx \bar{\rho} \\ \phi_+ & \text{for } \rho \gg \bar{\rho} \end{cases} \tag{4.6}$$

This definition will make much simpler to solve the integral of the bounce, which will give the preferred value of $\bar{\rho}$ when minimised.

First, we will simplify Eq. (4.3) by eliminating the $\rho^2\rho''$ term through integration by parts, which will give:

$$\begin{aligned} S_E =& 2\pi^2 \int \mathrm{d}\xi \left[\rho^3\left(\frac{1}{2}\phi'^2 + U\right) - \frac{3}{\kappa}(\rho\rho'^2 + \rho)\right] \\ =& 4\pi^2 \int \mathrm{d}\xi \left(\rho^3 U - 3\frac{\rho}{\kappa}\right), \end{aligned} \tag{4.7}$$

where in the second line I have made use of Eq. (4.5). This will be the action we will use to find the value of the coefficient $B = S_E - S_E(\phi_+)$.

Thus, at each region we obtain:

- **Outside:** Given at $\rho \gg \bar{\rho}$. At this region, the bounce and false vacuum are identical; thus,
$$B_{\text{out}} = 0 \tag{4.8}$$

- **Wall:** Given at $\rho \approx \bar{\rho}$. The solution will also correspond to a soliton; therefore,
$$B_{\text{wall}} = 2\pi^2\bar{\rho}^3 S_s, \tag{4.9}$$
where S_s is the action of a one-dimensional soliton with boundaries at the values of each vacuum in our potential.

- **Inside:** Given at $\rho \ll \bar{\rho}$. Note that the line element inside the bubble won't be the same as the one outside, because the curvature comes from the cosmological constant acting on the spacetime. As mentioned above, there will be different geometries at each region, and so we need to redefine ξ before evaluating the integral. By using Eq. (4.5), and noticing that ϕ is constant inside the bubble, we have that
$$\mathrm{d}\xi = \mathrm{d}\rho(1 - \frac{1}{3}\kappa\rho^2 U)^{-1/2}. \tag{4.10}$$
Therefore, the difference between the bounce and the stationary action is
$$\begin{aligned} B_{\text{in}} =& -\frac{12\pi^2}{\kappa}\int_0^{\bar{\rho}} \rho\,\mathrm{d}\rho \left[\left(1 - \frac{1}{3}\kappa\rho^2 U(\phi_-)\right)^{1/2} - (\phi_- \to \phi_+)\right] \\ =& \frac{12\pi^2}{\kappa^2}\left(U(\phi_-)^{-1}\left[\left(1 - \frac{1}{3}\kappa\bar{\rho}^2 U(\phi_-)\right)^{3/2} - 1\right] - (\phi_- \to \phi_+)\right). \end{aligned} \tag{4.11}$$

Adding the contribution from each region, we get the general expression for the coefficient B in the decay rate. In the original paper, Coleman and de Luccia study the transitions involving a Minkowski region, given that it was thought that our universe did not have a cosmological constant. However, as the aim of this essay is to understand the universe we live in, I will calculate the expression of $\bar{\rho}$ for a tunneling transition between two dS spacetimes.

As we did in QFT, now we just need to add the contributions from each region and find the value of ρ which minimizes it. After some algebra (shown in Appendix A), the stationary point of B is:

$$\frac{1}{\bar{\rho}^2} = \frac{1}{3}\kappa U(\phi_-) + \left(\frac{\varepsilon}{3S_s} + \frac{\kappa S_s}{4}\right)^2 \tag{4.12}$$
$$= \frac{1}{3}\kappa U(\phi_+) + \left(\frac{\varepsilon}{3S_s} - \frac{\kappa S_s}{4}\right)^2 \tag{4.13}$$

where ε is the difference of energy between both vacua.

Substituting this value for $\bar{\rho}$ in the equation for the coefficient B, we get:

$$B = \bar{\rho}^3 \left[2\pi^2 S_s - \frac{12\pi^2}{\kappa^2} U(\phi_+)^{-1} \left(\frac{\varepsilon}{3S_s} - \frac{\kappa S_s}{4} \right)^3 + \frac{12\pi^2}{\kappa^2} U(\phi_-)^{-1} \left(\frac{\varepsilon}{3S_s} + \frac{\kappa S_s}{4} \right)^3 \right] - \frac{12\pi^2}{\kappa^2} (U(\phi_+)^{-1} - U(\phi_-)^{-1}) \quad (4.14)$$

The behaviours of $\bar{\rho}$ and B encodes all the information about how gravity affects the bubble nucleation. The main difference in $\bar{\rho}$ is that it does not diverges in the limit of small ε. Moreover, from the expression on Eq. (4.12), it has a maximum value when the right-hand side is at its minimum. This happens when the parenthesis vanishes, and so when

$$\frac{\varepsilon}{3S_s} = \frac{\kappa S_s}{4}. \quad (4.15)$$

At this point, the value for $\bar{\rho}$ is

$$\bar{\rho} = \left(\frac{1}{3} \kappa U(\phi_+) \right)^{-1/2}, \quad (4.16)$$

which is not an arbitrary value, and its the most important requirement to maintain causality. As the decay takes place on a dS spacetime, there must be an event horizon. This is a causal limit, provided by the fact that the universe is expanding at an accelerated rate. Therefore, some regions of the spacetime move away faster than the speed of light, and so they are causally disconnected. The best way of calculating the event of horizon is by evaluating the general metric of a spacetime under the effect of a cosmological constant (Λ) in 4-dimensions:

$$\mathrm{d}s^2 = -f^2(\rho)\mathrm{d}t^2 + f^{-2}(\rho)\mathrm{d}\rho^2 + \rho^2 \mathrm{d}\Omega_2{}^2 \quad (4.17)$$

where

$$f(\rho)^2 = 1 - \rho^2 \frac{\Lambda}{3}. \quad (4.18)$$

and $\mathrm{d}\Omega_2$ is the line element of a two-sphere. By solving these equations, one can find that there is a coordinate singularity at $\rho = (3/\Lambda)^{1/2}$. It takes a particle finite proper time, but infinite coordinate time, to reach this radius[10]. Therefore, the event horizon must correspond to $\rho = (3/\Lambda)^{1/2}$.Thus, any two points which are further apart than this radius are causally disconnected.

The important bit comes from the fact that a scalar field at rest in a minimum creates an effective cosmological constant, given by

$$\Lambda = \kappa U(\phi_o). \quad (4.19)$$

where

$$U'(\phi_o) = 0 \quad \text{and} \quad \frac{\mathrm{d}\phi}{\mathrm{d}t}|_{\phi=\phi_o} = 0.$$

Therefore, we find that the event horizon created by a scalar field resting at its minimum is $\rho = (3/\kappa U(\phi_o))^{-1/2}$ which is the same result found at Eq. (4.15).
This is a very important statement, given that the event horizon delimits the size of the universe. There cannot be bigger bubbles nucleated because they would not fit in the universe they live in. Moreover, if they were nucleated at a higher radius, causality would be broken. This is because a change of vacua does transmit information, and so it could be used to contact parts of the universe to which light cannot travel to; and therefore travelling faster than the speed of light.

However, once the bubble is nucleated, the geometry of the spacetime will change. Therefore, to stablish a coordinate system on the spacetime, we need to stop working with imaginary time. In QFT, after calculating the bounce in Euclidean space we could come back to Minkowski spacetime by analytic continuation. This was straightforward because the shape of the space was constant before and after the tunneling, so we just needed to undo the rotation to imaginary time. However, when gravity is considered, the energy of the scalar field does change the geometry of the spacetime. Therefore, we expect the metric inside the bubble to be different to the one outside.

4.2 Coming back to reality

As stated before, SO(4) symmetry of the bounce is maintained when gravity is included. Therefore, the bounce solution does not only affects the tunneling rate of the system, but also predicts the creation of a true vacuum bubble inside a false vacuum background.

We expect the bubble to expand as in the flat space case, where the speed of the walls will asymptotically reach the speed of light. This will lead up to the point where the bubble occupies all the region inside the future lightcones, as in Fig.4. However, in this case the light-cones will have a different behaviour inside the universe, given that their path will be influenced by the geometry of the spacetime.

For this, I will approximate the bubble to have the size of the future causal lightcone. Therefore, the regions inside and outside the lightcones of the origin (i.e. the centre of the bubble) must be treated separately. Outside the lightcone, the space is causally disconnected to the bubble, so we expect it to keep on being a dS spacetime with metric:

$$\mathrm{d}s^2 = \mathrm{d}\xi^2 + \rho(\xi)^2 \mathrm{d}\Omega_S{}^2, \tag{4.20}$$

where the Euclidean

$$\mathrm{d}\Omega_3{}^2 = \mathrm{d}\tau^2 + \cosh^2\tau \mathrm{d}\Omega_2{}^2$$

has been replaced by the Lorentzian

$$\mathrm{d}\Omega_S{}^2 = -\mathrm{d}\tau^2 + \cosh^2\tau \mathrm{d}\Omega_2{}^2,$$

which is the metric on a unit hyperboloid with spacelike normal vector.

In this coordinates, $\rho = 0$ stops being a point, and it delimits the whole hypersurface defined by the lightcones of the nucleation point. Therefore, we see that these coordinates do not cover the entire spacetime, but just the region outside the bubble.

Therefore, we must extend the coordinate system to inside the bubble. From the properties of this region, we know that it is a finite extension of space that is expanding due to the presence of the true vacuum energy. Moreover, the walls are moving outwards at the speed of light. This means that any observer living inside this bubble can only reach assymptotically the moving wall. Therefore, the best fit for this spacetime is an open Friedmann-Robertson-Walker universe,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \left(\frac{\mathrm{d}r^2}{1+r^2} + r^2 \mathrm{d}\Omega^2 \right), \tag{4.21}$$

where $a(t)$ is the scale factor that obeys the Friedmann equation:

$$\dot{a}^2 = 1 + \frac{\kappa}{3} a^2 \left(\frac{1}{2}\dot{\phi}^2 + V \right) \tag{4.22}$$

(with $\dot{} = \frac{\mathrm{d}}{\mathrm{d}t}$) and the scalar field satisfies:

$$\ddot{\phi} + \frac{3\dot{a}}{a}\dot{\phi} = -\frac{\mathrm{d}V}{\mathrm{d}\phi}. \tag{4.23}$$

These are the MPEP equations for the classically allowed region, as expected. If the system tunnels directly to the true vacuum, ϕ_-, the scale factor will describe a dS universe, so $a(t) = H_-^{-1} \sinh H_- t$ (where H_- is the Hubble parameter for a dS universe at ϕ_-). Inside the bubble, there is no such a thing as a preferred location, and so the space is homogeneous and isotropic. It is very important that the interior and exterior metrics join smoothly across the boundary, which is given at the zero of a.

4.2.1 Picture a bubble

To better understand the geometry of the universe after a bubble materializes, it is very useful to use Penrose diagrams. For that, we must find a compact set of coordinates for de Sitter space.

Due to the SO(4,1) symmetry group of dS, it can be embedded in a 5-Dimensional Minkowski space time. Taking coordinates X_μ for $\mu = 0, ..., 4$, then the metric can be written as

$$\mathrm{d}s^2 = \eta_{\mu\nu} X^\mu X^\nu.$$

Now we just need to define the X_μ coordinates to satisfy the conformally compact properties of Penrose diagrams. A useful choice is

$$X_0 = \tan T \tag{4.24}$$

$$X_i = \frac{\sin\eta}{\cos T}\omega_i \quad \text{for } i = 1, 2, 3 \tag{4.25}$$

$$X_4 = \frac{\cos\eta}{\cos T}, \tag{4.26}$$

where $(\omega_1, \omega_2, \omega_3) = (cos\theta, \sin\theta\cos\phi, \sin\theta\cos\phi)$ stores the angular coordinates. The de Sitter metric in these coordinates is

$$\mathrm{d}s^2 = \frac{1}{\cos^2 T}[-\mathrm{d}T^2 + \mathrm{d}\eta^2 + \sin^2\eta \mathrm{d}{\Omega_2}^2], \tag{4.27}$$

with $-\pi/2 \leq T \leq \pi/2$ and $0 \leq \eta \leq \pi$ being the temporal and radial coordinates, respectively.

The Penrose diagram of this metric can be seen in Fig.6. In this picture, every vertical grey line corresponds to timelike geodesics, while the horizontal ones define the surfaces of constant conformal time. As we can tell, they do not agree everywhere, given the different rates of expansion. However, every light ray goes at 45^o in both regions of space. This is the reason why the space inside the bubble protrudes the conformal square of the background. If we want both regions to have the same null geodesics, then we can only compact one of them. However, this lets us join both regions smoothly through the domain wall of the bubble, which we can tell that expands at the speed of light.

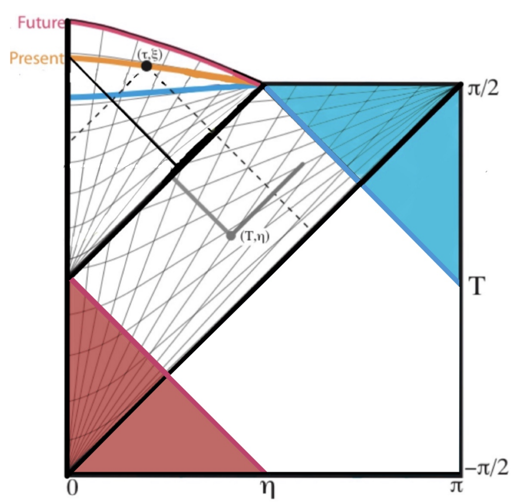

Figure 6: Penrose diagram of the nucleation of a bubble inside a de Sitter background. An observer at position (τ, ξ) inside the bubble could see a collision from a bubble that nucleated at (T, η). Bubbles nucleated at the blue region, would not reach our bubble due to expansion. Similarly, bubbles nucleated in the red region, would change the vacua of $\eta = 0$ before our bubble is nucleated, and so tunneling would not take place, breaking the logic of this Penrose Diagram.

Moreover, we know that in dS spacetime two points can be causally disconnected if their future lightcones do not intersect. Therefore, if a second bubble materializes at the blue region in the diagram from Fig.6, it won't collide with the first one. But if a second bubble nucleates in the white region, there will be a collision with the first one, leaving imprints that can be observed from inside.
However, the rate of collisions will be affected by expansion because of the presence of horizons. If the nucleation rate is very high, then bubbles will appear at a faster rate than the universe expands, dominating the whole space and eventually giving rise to a true vacuum universe. In the case where the nucleation rate is small, space would expand faster than bubbles materialise, never reaching a full dominance of the true vacuum.

The next logical step is to wonder what would we see if we were living inside one of these bubbles. Therefore, in the next section I will talk about the possibility of observing these collisions from inside our universe.

5 Our Universe and Bubble Collisions

The proposal that our universe comes from a tunneling process was originally discarded because no generic potential as the ones shown above could fit the entropy density in our universe [8]. However, it was found that a non-generic potential could give very different results[11][12]. If $U(\phi)$ has a flat region between the potential barrier and the true vacuum minimum at ϕ_- (Fig.7), then the slow-roll of scalar field could make the universe inside the bubble to grow exponentially before the field comes at rest at the minimum, agreeing with current observations.

The first attempts to predict anisotropies from bubble collisions in our universe was done by Guth *et al.*[13]. In their paper, they assumed the rate of expansion inside and outside the bubble to be the same. However, three years later, the calculations were repeated for different rates of expansion at each region [14]. In this section, I will follow their work to show the expected number of collisions that we should see in our night sky for a potential such as Fig. 7. Moreover, I will focus on the case in which our potential has more than one minimum to which the system can nucleate to. This is a picture that better agrees with fundamental theories such as String Theory.

To make calculations on our universe I will use the picture of the space derived in last section, and shown in Fig.6. The dS metric outside the bubble has an event horizon radius of H_f^{-1}, while the space inside the bubble will begin in slow-roll inflation with an event horizon radius of H_i^{-1}. Using the Friedman equations, $H_{f,i}$ is related to the potential energy of the field by

$$H_{f,i}^2 = \frac{\kappa}{3} U(\phi_{f,i}), \tag{5.1}$$

where $\kappa = 8\pi G$ and $\phi_{f,i}$ are the values for which the system is at the false vacuum and at the slow roll period in the potential, respectively.

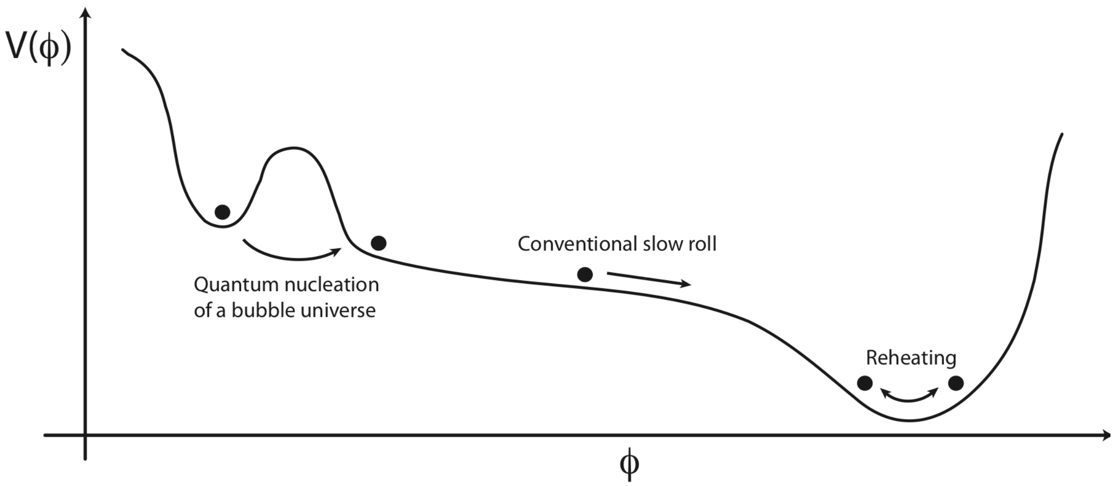

Figure 7: General potential energy that could describe the history of our universe from tunneling transitions. Credit [15]

A convenient coordinate system that covers the region where collision bubbles can nucleate is

$$\mathrm{d}s^2 = H_f^{-2}\frac{1}{\cosh^2\chi}(\mathrm{d}\chi^2 - \mathrm{d}\tau^2 + \cosh^2\tau \mathrm{d}{\Omega_2}^2) \tag{5.2}$$

which is equivalent to the metric on Eq. (4.20), where ξ solves the equations

$$\frac{\mathrm{d}\xi}{\rho(\xi)} = \mathrm{d}\chi, \qquad \rho(\chi) = \frac{1}{H_f\cosh\chi}$$

Therefore, χ corresponds to the radial distance and τ to the conformal time. They both run from $-\infty$ to ∞. Equivalently, $\chi = \infty$ bounds the region covered by this metric.
After a second bubble is nucleated at a given time τ, the distance between both bubbles will change the behaviour of the collision. Therefore, the background geometry will have a SO(2,1) symmetry. Thus, rotations around the radial curve connecting both bubbles will leave the system invariant.

Inside the bubble, the metric is given by Eq. (4.21). Therefore, it can be described by an homogenous and isotropic open FRW metric:

$$\mathrm{d}s^2 = a^2(\eta)(-\mathrm{d}\eta^2 + \mathrm{d}\rho^2 + \sinh^2\eta \mathrm{d}{\Omega_2}^2) \tag{5.3}$$

where $a(\eta)$ is the scale factor, η is the conformal time inside the bubble and without loss of generality we will focus on an observer at $\rho = 0$. As stated in last section, this metric must be smooth all the way up to the big bang, where $a(\eta) \to 0$.

Therefore, as it has to agree with the background vacuum at the nucleation time, we expect the scale factor to decrease as

$$a(\eta) \approx \frac{2}{H_f} e^{\eta} \quad \text{as} \quad \eta \to -\infty \tag{5.4}$$

where this expression comes from the fact that in a dS space-time, $a(\tau) = H_f^{-1} \sinh H_f \tau$. Now we can parametrize null rays propagating from outside to inside the bubble.

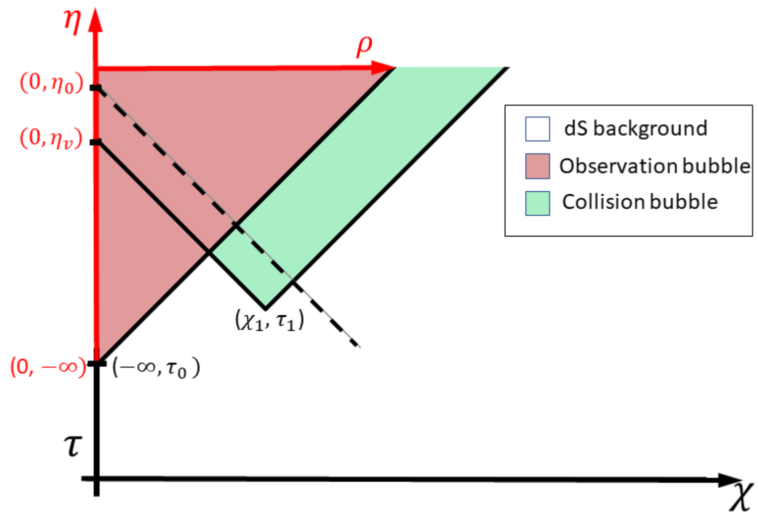

Figure 8: Diagram of the different regions of space once two bubbles are nucleated at $(-\infty, \tau_0)$ and (χ_1, τ_1). The black coordinates, (χ, τ), and white region corresponds to the background de Sitter, while red corresponds to the coordinates from inside the bubble, (ρ, η). Light rays are set to move at 45^o inside every region. The dashed line corresponds to the causal past of an observer sitting in the centre of the bubble, $(0, \eta_0)$.

Consider a bubble that nucleates at some point in $(\chi, \tau, \theta, \phi)$. The angular location will be given by (θ, ϕ). Ingoing radial null curves propagating outside the bubble satisfy $\chi + \tau = c$. Similarly, inside the bubble they satisfy $\eta + \rho = c$, where c is an arbitrary constant. As we want to compute the time it takes the collision to enter the backward lightcone of the observer at $\rho = 0$, we have that

$$\eta_v = \chi + \tau \tag{5.5}$$

where η_v is the conformal time at which the the collision first becomes visible, as in Fig.8. The probability to nucleate a bubble in an infinitesimal region is proportional to the 4-volume of that region,

$$\mathrm{d}N = \Gamma \mathrm{d}V_4 \tag{5.6}$$

where Γ is the nucleation rate. By using the background system defined at Eq. (5.1) and defining the dimensionless nucleation rate $\gamma = H_f^{-4}\Gamma$, we get

$$\mathrm{d}N = \gamma \frac{\cosh^2 \eta_v - \chi}{\cosh^4 \chi} \mathrm{d}\chi \mathrm{d}\eta_v \mathrm{d}^2\Omega_2 \tag{5.7}$$

where I substituted the equality of Eq. (5.4) on τ. We might expect the integral to be evaluated in the range $-\infty < \eta_v < \eta_o$ (being η_o the present conformal time), which would give an infinite number of bubble nucleated. However, the fact of having two different vacua inside each bubble will create a domain wall propagating at almost the speed of light inside the observation bubble. It is important to make calculations which are consistent with the presence of observers, and so we expect this domain wall to go away from the observation bubble after the collision. This is because it could disrupt structure formation throughout its future lightcone.

To parametrize this, we adopt the following picture of the domain wall motion: we assume the wall moves into the observation bubble at the speed of light for a time of order H_f^{-1}. Then, it bounces back after the collision, and moves away at the speed of light.
In the choice of coordinates we have been using, $\eta = 0$ corresponds to a time of order H_f^{-1}. Therefore, we will approximate the domain wall trajectory as null and ingoing for $\eta < 0$, and null and outgoing for $\eta > 0$. Given the assumptions, we must evaluate the integral in the range $0 < \eta_v < \eta_o$, which gives:

$$N(\eta_o) = \frac{8\pi\gamma}{3}(\sinh 2\eta_o + \eta_o). \tag{5.8}$$

Now, we must look for an expression of the conformal time, η_o, for an observer inside one of these bubbles. For that, following Weinberg's book [16], the expansion of our universe can be mainly divided in two parts:

- **Slow-roll inflation:** Given at times $\eta < \eta_{RH}$, where RH denotes for reheating. We can approximate the slow-roll inflation by de Sitter space with Hubble constant H_i. Recalling the definition of the conformal time, $\mathrm{d}\eta = \mathrm{d}t/a(t)$, being t the proper time of the observer, we can express the scale factor as a function of conformal time as

$$a^2(\eta) = \frac{1}{H_i^2 \sinh^2 \eta - \log \frac{H_f}{H_i}} \tag{5.9}$$

  where the $\log H_f/H_i$ comes from the boundary condition imposed by Eq. (5.3). Defining the number of e-foldings (N) at reheating by $a_{RH} \equiv H_i^{-1} e^N$, the conformal time at reheating is

$$\eta_{RH} = \log \frac{H_f}{H_i} - e^{-N} \approx \log \frac{H_f}{H_i} \tag{5.10}$$

  for $a_{RH} \gg H_i$ and $N \gg 1$.

- **After reheating:** Given at times $t > t_{RH}$. At this stages, the behaviour of the scale factor depends upon which type of matter is dominating the expansion of the universe at a particular epoch. In this sense, the scale factor would evolve as

$$a(t) = H_i^{-1} e^N (\frac{t}{t_{RH}})^p \tag{5.11}$$

  where $p = 1/2$ or $2/3$ for radiation or matter domination respectively.

Integrating this expression to get the conformal time, we obtain:

$$\eta(t) - \eta_{RH} = \frac{1}{1-p}\left(\frac{t}{a(t)} - \frac{t_e}{a(t_e)}\right) \tag{5.12}$$

which for times much larger than reheating and using the Friedman's equation, we get to the expression

$$\eta_o \approx \log\frac{H_f}{H_i} + \zeta_o\sqrt{\Omega_k} \tag{5.13}$$

where ζ_o is the effective contribution from each dominating fluid over time and Ω_k is the curvature contribution to the total energy density. Substituting this into the expression for the total number of collisions (Eq. (5.7)), we find

$$N \approx \frac{4\pi\gamma}{3}\left[\left(\frac{H_f}{H_i}\right)^2 e^{2\zeta_o\sqrt{\Omega_k(t)}} + 2\log\frac{H_f}{H_i} + 2\zeta_o\sqrt{\Omega_k(t)}\right]. \tag{5.14}$$

Given that our in our universe $\Omega_k(t_0) \ll 1$ and also $H_f/H_i \gg 1$, we get

$$N \approx \frac{4\pi\gamma}{3}\left(\frac{H_f}{H_i}\right)^2. \tag{5.15}$$

While one expects $\gamma \ll 1$, observational constraints require $(H_f/H_i)^2 \geq 10^{12}$. Therefore, there is a very wide range of fine-tuning that can satisfy the physical observations.
This result creates a very useful framework for future calculations. From this, we learn that the number of expected collisions we must observe depends on the rate of expansion of our universe and the false vacuum background. Moreover, any observer living inside one of this bubble universe is able to see collisions at some point, which is a prediction of the eternal inflation theories with multiple minima, such as String Theory [17].

The calculation presented in this section is only the starting point of this area of study. During the whole essay, I have been trying to work model independently, which is useful for the qualitative picture of the evolution of the universe. However, when we take less assumptions and for instance we stop working in the thin wall limit, the equations become very model dependent. On one side, it makes all the calculations very difficult and tedious to solve, so finding an analytic expression of the solution is almost impossible task to do. However, on the other side, by using simulations we can get very good estimations of the effect that certain potentials can have on our observations, which will vary notably depending on the model. A lot of advancement has been done in this field in the last years, see references [18][19][20].
Given that the solution is model dependant, it will be easy to discard wrong potentials from observations once we have our predictions. Therefore, by comparing the simulations with reality, we will start constraining the range of valid potentials that can be driving the expansion of the universe.

6 Conclusion

This essay has presented an analysis of the evolution of a dynamical scalar field with two different minima, one being the global minimum. Even though the idea might seem simple, we found the implications that these kind of systems could have on our universe. Giving us an accurate description of the shape of our spacetime and having big implications on the validity of String Theory.

We started by describing how quantum mechanics changed the whole understanding that classical physics had from this kind of potentials. From there, we found that quantum tunneling breaks the stability of all the non-global minima, making it to be unstable at long times. This was generalised to Quantum Field Theory, given that it can be described as a quantum mechanical system of infinity dimensions.
For low energy difference, we found that with each tunneling, there is a bubble of true vacuum being nucleated. Depending on where the system tunnels to, it can materialise with different radius. Once it comes to existence, the bubble starts expanding with a speed which increases asymptotically to the speed of light.

It was very important to consider gravity given that a scalar field can change the geometry of the spacetime. A positive potential energy creates an expanding de Sitter universe, while a negative energy would create an anti-de Sitter universe. Similarly, for vanishing energy, we recover Minkowski geometry, and so calculations would agree with those from QFT.
From this, we found that inside the bubble there will form an open FRW universe, surrounded by the original dS background. Moreover, there is a maximum bubble radius, given by the event horizon of the background. Depending on the rate of nucleation, we might expect collisions of different bubbles. Therefore, in the last section we tried to describe the effect of these collisions for an observer inside one of the bubbles.
In this picture, we have an expanding dS universe inside the bubble, which would correspond to our universe. After finding a suitable coordinate system, we calculated the number of expected collisions that an observer inside the bubble can measure at a given conformal time, η_o. After calculating the estimated proper time for our universe, we found:

$$N \approx \frac{4\pi\gamma}{3}\left(\frac{H_f}{H_i}\right)^2. \tag{6.1}$$

Depending on the properties of the impact, the collision can create from small anisotropies in the CMB to big temperature patches on our night-sky.
The best predictions have been done via numerical methods, which will let us compare simulations with our own universe. This will have a big impact on Sting Theory, which predicts a potential with multivalued minima. Moreover as these collisions depend on the nucleation rate and the expansion of the false vacuum, we can use our predictions to start constructing the shape of the effective potential that is driving the expansion of our universe.

However, to have a deeper and more complete understanding of tunneling, we must consider two important aspects from quantum mechanics not covered in this essay: resonant tunneling and thermal effects. Even though it has been shown that resonant tunneling cannot exist on QFT [5], it would be worth studying how gravity could affect this result. Moreover, it is well known that a dS universe has a finite temperature from quantum effects, the so called Gibbons-Hawking temperature [21]. This could create a thermal nucleation of bubbles [22], which would not materialize through tunneling. However, there can be a combination of these two effects to give thermal assisted tunneling [23], which would influence the calculations presented in this essay.
Moreover, even though we just worked on dS→dS tunneling, we can apply these techniques to different spacetimes. For instance, we can stop working with maximal symmetric spaces by considering the presence of matter, which will have big implications on strong gravity limits, such as black holes. However, to study these implications, it is convenient to use different techniques to the ones used in this report, which come directly from String Theory [24].

Even though there is much left to discover, this essay has presented one of the most important ideas of cosmology. The main axioms of modern cosmology are that the laws of physics are homogeneous and isotropic. However, the existence of structures shows that these symmetries broke at some point in our universe. So far, we have a good understanding on how these anisotropies formed, but we still treat the expansion to be homogeneous everywhere. This makes sense, given that we do not have enough evidence to think otherwise. However, recent studies seem to be pointing to the possibility of this homogeneity to be also broken [25]. Although there is no scientific consensus on these results, we shall wonder how this effect might be produced just in case some day we find it to be the case. For this, the conceptual ideas of how to study the different regions of expansion will be the same to the ones shown in this essay. For every geometry of the universe, we can find a non-homogeneous scalar field that might be driving the different rates of expansion.
Finding that the expansion of the universe is not homogeneous and isotropic would change the whole picture of modern cosmology. However, the work presented in this paper would be key to understand the properties of the spacetime.

The problem of the expansion of our universe has been in the frontline of physics for decades, and getting to the truth of the inflation is a very hard process that might take a few decades more. Until there, the focus of cosmology should be in the study of which conditions we need the last theory to satisfy, and so far, we have found that scalars fields give a very good description. To be strict with our analysis, we must bare in mind that we live in a quantum-like universe, and so we shall expect scalar fields to also behave in this manner, leading up to the creation of expanding bubbles of different vacua. Further measurements and research will give better constraints on the description of the possible potentials driving our expansion, placing us a step closer to the fundamental nature of our universe.

Appendices

A $\bar{\rho}$ in dS→dS tunneling

As stated in Section 4, I closely follow the derivations from Weinberg [8]. However, the calculation of the expression for $\bar{\rho}$ is not included in his book. Therefore, I thought of including it in this paper to show a direct proof of this result.

To find the expression for $\bar{\rho}$, we just need to minimize the B coefficient. Summing over the contributions from each region, we get the expression

$$B = 2\pi^2\bar{\rho}^3 S_s + \frac{12\pi^2}{\kappa^2}\left(U(\phi_-)^{-1}\left[\left(1-\frac{1}{3}\kappa\bar{\rho}^2U(\phi_-)\right)^{3/2}-1\right]-(\phi_-\to\phi_+)\right).$$

Then, differentiating with respect to $\bar{\rho}$:

$$\frac{\partial B}{\partial\bar{\rho}} = 0 = 6\pi^2\bar{\rho}^2 S_s + \frac{12\pi^2}{\kappa^2}\left[\kappa\bar{\rho}\left(1-\frac{1}{3}\kappa\bar{\rho}^2U(\phi_-)\right)^{1/2}-(\phi_-\to\phi_+)\right].$$

Now, we just need to solve this equation. For that, it will be convenient to use the following redefinitions

$$\Lambda_\pm = \frac{1}{3}\kappa U(\phi_\pm), \qquad\qquad \bar{S} = \frac{\kappa}{2}S_S.$$

With this, the above expression takes the form:

$$-\bar{\rho}^2\bar{S} = \bar{\rho}[(1-\bar{\rho}^2\Lambda_-)^{1/2}-(1-\bar{\rho}^2\Lambda_+)^{1/2}].$$

Squaring at both sides and moving the $\bar{\rho}^2$ factors to the left hand side (LHS) we obtain

$$\bar{\rho}^2(\bar{S}^2+\Lambda_++\Lambda_--2\bar{\rho}^{-2}) = -2(1-\bar{\rho}^2\Lambda_-)^{1/2}(1-\bar{\rho}^2\Lambda_+)^{1/2}.$$

To remove square roots of the RHS, we will square at both sides once again. Treating each side separately will make the calculation to be less messy. Then,

$$\begin{aligned}\text{LHS} =& \bar{\rho}^4(\bar{S}^2+\Lambda_++\Lambda_-)^2\\ =& \bar{\rho}^4\bar{S}^4+\bar{\rho}^4(\Lambda_++\Lambda_-)^2+4-4\bar{\rho}^2(\Lambda_-+\Lambda_+)+2\bar{S}^2\bar{\rho}^4(\Lambda_-+\Lambda_+-2\bar{\rho}^{-2}),\end{aligned}$$

$$\text{RHS} = 4-4\bar{\rho}^2(\Lambda_-+\Lambda_+)+4\bar{\rho}^4\Lambda_-\Lambda_+.$$

Cancelling equal terms ant both sides and knowing that $(a+b)^2 - 4ab = (a-b)^2$, we find

$$0 = \bar{\rho}^4 \bar{S}^4 + \bar{\rho}^4(\Lambda_+ - \Lambda_-)^2 + 2\bar{S}^2\bar{\rho}^4(\Lambda_- + \Lambda_+ - 2\bar{\rho}^{-2})$$

Then, defining $\Delta\Lambda = \Lambda_+ - \Lambda_-$

$$\frac{4}{\bar{\rho}} = \bar{S}^2 + \Delta\Lambda^2\bar{S}^{-2} + 2(\Lambda_- + \Lambda_+)$$

We can complete the square in the RHS either using Λ_- or Λ_+, which will give two different solutions:

$$\begin{aligned}\frac{1}{\bar{\rho}^2} =& \Lambda_- + \left(\frac{\Delta\Lambda}{2\bar{S}} + \frac{\bar{S}}{2}\right) \\ =& \Lambda_+ + \left(\frac{\Delta\Lambda}{2\bar{S}} - \frac{\bar{S}}{2}\right).\end{aligned}$$

Finally, undoing back the redefinitions, we obtain the desired result:

$$\begin{aligned}\frac{1}{\bar{\rho}^2} =& \frac{1}{3}\kappa U(\phi_-) + \left(\frac{\varepsilon}{3S_s} + \frac{\kappa S_s}{4}\right)^2 \\ =& \frac{1}{3}\kappa U(\phi_+) + \left(\frac{\varepsilon}{3S_s} - \frac{\kappa S_s}{4}\right)^2,\end{aligned}$$

where $\varepsilon = U(\phi_+) - U(\phi_-)$.

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